AP/IB Exam Review

Key

AP Calculus AB Mini Lesson on Derivatives

Why are derivatives necessary?

- Slope of tangent lines
- Instantaneous velocity
- Distance, velocity, acceleration
- Rate of change

The Product Rule

$$h(x) = f(x) \cdot g(x)$$

$$h'(x) = f'g + g'f$$

The Quotient Rule

$$h(x) = \frac{f(x)}{g(x)} \quad \text{where } g(x) \neq 0$$

$$h'(x) = \frac{f'g - g'f}{g^2}$$

The Chain Rule $h(x) = (3x+5)^4$

$u = 3x+5$

$u' = 3$

$v = u^4$

$v' = 4u^3$

$h'(x) = u' \bullet v'$

$h'(x) = 3 \bullet 4(3x+5)^4$

$h'(x) = 12(3x+5)^4$

Implicit Differentiation

b. $x^2 - 2y^3 + 4y - 2 = 0$

$$2x - 6y^2\left(\frac{dy}{dx}\right) + 4\left(\frac{dy}{dx}\right) = 0$$

$$-6y^2\left(\frac{dy}{dx}\right) + 4\left(\frac{dy}{dx}\right) = -2x$$

$$\frac{dy}{dx}\left(-6y^2 + 4\right) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{-6y^2+4}$$

Definition of critical number:

Let f be defined at c.

If f '(c) = 0 OR if f ' is **undefined** at c,

then c is *critical number* of f

Relative Extrema Occur Only at Critical Numbers:

If f has a relative minimum or maximum at x = c,

then c is critical number of f.

Not all critical numbers produce extrema.

First Derivative Test

Let c be a critical number of a function f that is continuous on an open interval containing c.

If f is differentiable on the interval, except possibly at c, then f(c) can be classified as follows:

1) If f'(x) changes from NEG to POS at c, then f(c) is a relative MIN
2) If f'(x) changes from POS to NEG at c, then f(c) is a relative MAX

Test for Concavity (The 2nd Derivative Test):

Let f be a function whose second derivative exists on the open interval.

1. If f "(x) > 0 for all x, then the graph of f(x) is concave upward

2. If f "(x) < 0 for all x, then the graph of f(x) is concave downward

The Slope of the SECANT line is the average rate of change

The Slope of the TANGENT line is the instantaneous rate of change

<u>To find the equation of a tangent line to f(x) at x = a:</u>

1) Find f ' (x)
2) Evaluate f '(a) to determine m.
3) Evaluate f(a) to determine y-coordinate.
4) $y - y_1 = m(x - x_1)$

When f'(x) = 0 or f'(x) is undefined, you have critical points

Some Critical Points are extrema

When f'(x) > 0 the function f(x) is increasing
(derivative is positive)
When f'(x) < 0 the function f(x) is decreasing
(derivative is negative)

When f''(x) = 0 , you have a possible POI

When f''(x) > 0 the function f(x) is concave up
(2nd der. is positive)
When f''(x) < 0 the function f(x) is concave down
(2nd der. is negative)

If f'(x) = 0 and f''(x) < 0, the function f(x) has a maximum
If f'(x) = 0 and f''(x) > 0, the function f(x) has a minimum

Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I.

If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0$$

Non - Calculator MC

3. If $f(x)=(x-1)(x^2+2)^3$, then $f'(x)=$ $(x^2+2)^3 + 3(x^2+2)^2(2x)(x-1)$

$(x^2+2)^2(x^2+2+6x^2-6x)$

$(x^2+2)^2(7x^2-6x+2)$

(A) $6x(x^2+2)^2$

(B) $6x(x-1)(x^2+2)^2$

(C) $(x^2+2)^2(x^2+3x-1)$

(D) $(x^2+2)^2(7x^2-6x+2)$

(E) $-3(x-1)(x^2+2)^2$

16. If $\sin(xy)=x$, then $\frac{dy}{dx}=$

$\cos(xy)\left[y+\frac{dy}{dx}\cdot x\right]=1$

$y\cos(xy)+x\cos(xy)\frac{dy}{dx}=1$

$\frac{1-y\cos(xy)}{x\cos(xy)}=\frac{dy}{dx}$

(A) $\frac{1}{\cos(xy)}$

(B) $\frac{1}{x\cos(xy)}$

(C) $\frac{1-\cos(xy)}{\cos(xy)}$

(D) $\frac{1-y\cos(xy)}{x\cos(xy)}$

(E) $\frac{y(1-\cos(xy))}{x}$

$f(x)=e^{2x^{-1}}$

12. If $f(x)=e^{(2/x)}$, then $f'(x)=$ $\left(e^{\frac{2}{x}}\right)\left(-2x^{-2}\right)=\frac{-2e^{2/x}}{x^2}$

(A) $2e^{(2/x)}\ln x$ (B) $e^{(2/x)}$ (C) $e^{(-2/x^2)}$ (D) $-\frac{2}{x^2}e^{(2/x)}$ (E) $-2x^2e^{(2/x)}$

x	0	1	2	3
$f''(x)$	5	0	−7	4

14. The polynomial function f has selected values of its second derivative f'' given in the table above. Which of the following statements must be true?

(A) f is increasing on the interval $(0, 2)$.

(B) f is decreasing on the interval $(0, 2)$.

(C) f has a local maximum at $x=1$.

(D) The graph of f has a point of inflection at $x=1$.

(E) The graph of f changes concavity in the interval $(0, 2)$.

Non - Calculator MC

20. Let f be a function with a second derivative given by $f''(x) = x^2(x-3)(x-6)$. What are the x-coordinates of the points of inflection of the graph of f?

$x = 0, 3, 6$

2nd Deriv Test

POS, POS, NEG, POS

0 3 6

(A) 0 only

(B) 3 only

(C) 0 and 6 only

(D) 3 and 6 only

(E) 0, 3, and 6

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24. The function f is twice differentiable with $f(2)=1$, $f'(2)=4$, and $f''(2)=3$. What is the value of the approximation of $f(1.9)$ using the line tangent to the graph of f at $x=2$?

(A) 0.4 (B) 0.6 (C) 0.7 (D) 1.3 (E) 1.4

point: $(2, 1)$

$m_{tan} = 4$

$y - 1 = 4(x-2)$

$y - 1 = 4(1.9-2)$

$y - 1 = 4(-.1)$

$y = -.4 + 1$

$\boxed{y = .6}$

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28. Let f be a differentiable function such that $f(3)=15$, $f(6)=3$, $f'(3)=-8$, and $f'(6)=-2$. The function g is differentiable and $g(x)=f^{-1}(x)$ for all x. What is the value of $g'(3)$?

$f^{-1}(3) = 6 = g(3)$

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(6)}$$

$$= -\frac{1}{2}$$

(A) $-\frac{1}{2}$

(B) $-\frac{1}{8}$

(C) $\frac{1}{6}$

(D) $\frac{1}{3}$

(E) The value of $g'(3)$ cannot be determined from the information given.

Calculator MC

78. The first derivative of the function f is defined by $f'(x)=\sin(x^3-x)$ for $0\le x\le 2$. On what interval(s) is f increasing?

$0=\sin(x^3-x)$

$x=1,\ x=1.691$

First Deriv Test

[NEG | POS | NEG]

0   1   1.691   2

(A) $1\le x\le 1.445$

(B) $1\le x\le 1.691$

(C) $1.445\le x\le 1.875$

(D) $0.577\le x\le 1.445$ and $1.875\le x\le 2$

(E) $0\le x\le 1$ and $1.691\le x\le 2$

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80. The derivative of the function f is given by $f'(x)=x^2\cos(x^2)$. How many points of inflection does the graph of f have on the open interval $(-2, 2)$?

↳ when f' changes inc. to dec. or dec to inc.

(A) One   (B) Two   (C) Three   (D) Four   (E) Five

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82. A particle moves along a straight line with velocity given by $v(t)=7-(1.01)^{-t}$ at time $t\ge 0$. What is the acceleration of the particle at time $t=3$?

(A) -0.914 (B) 0.055   (C) 5.486   (D) 6.086   (E) 18.087

MATH #8 : nDeriv(v(t), x, 3)

or Graph it and go to 2nd Calc #6 and enter 3 for X

90. The function f is continuous on the closed interval $[2, 4]$ and twice differentiable on the open interval $(2, 4)$. If $f'(3) = 2$ and $f''(x) < 0$ on the open interval $(2,4)$, which of the following could be a table of values for f? ↳ f' (ie slopes) should decrease

(A)

x	$f(x)$
2	2.5
3	5
4	6.5

(B)

x	$f(x)$
2	2.5
3	5
4	7

(C)

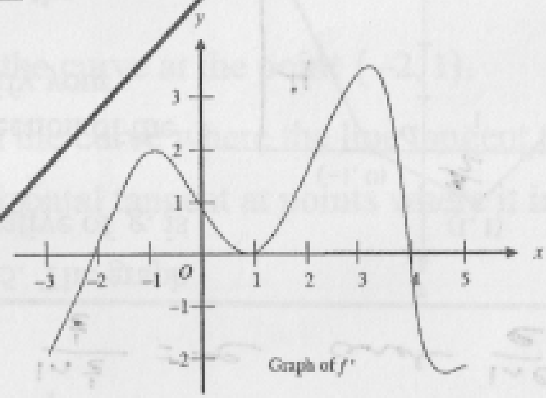
x	$f(x)$
2	3
3	5
4	6.5

(D)

x	$f(x)$
2	3
3	5
4	7

(E)

x	$f(x)$
2	3.5
3	5
4	7.5



84. The graph of the derivative of a function f is shown in the figure above. The graph has horizontal tangent lines at $x = -1$, $x = 1$, and $x = 3$. At which of the following values of x does f have a relative maximum?

(A) −2 only

(B) 1 only

(C) 4 only

(D) −1 and 3 only

(E) −2, 1, and 4

Key attached for FRQs

Non-Calculator FR

#1 Let f be the function given by $f(x) = \dfrac{\ln x}{x}$ for all $x > 0$. The derivative of f is given by

$$f'(x) = \frac{1 - \ln x}{x^2}.$$

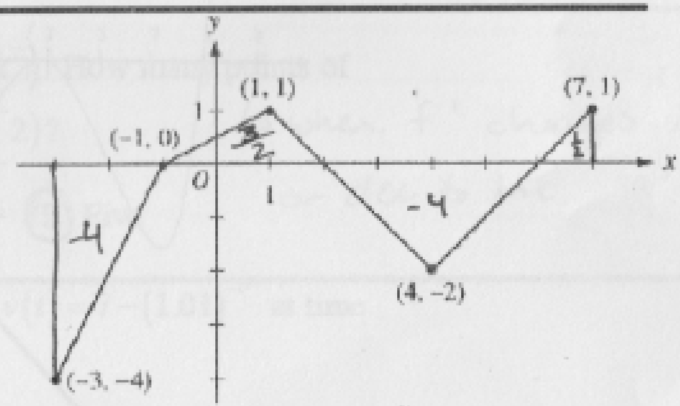
(a) Write an equation for the line tangent to the graph of f at $x = e^2$.

(b) Find the x-coordinate of the critical point of f. Determine whether this point is a relative minimum, a relative maximum, or neither for the function f. Justify your answer.

(c) The graph of the function f has exactly one point of inflection. Find the x-coordinate of this point.

(d) Find $\lim\limits_{x \to 0^+} f(x)$. $= \boxed{-\infty}$ → Show $\dfrac{\ln \frac{1}{2}}{\frac{1}{2}} = -e$ and $\dfrac{\ln e}{e} = \dfrac{1}{e}$

#2 Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.



Graph of g'

(a) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

(b) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

(c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

Key attached

Non-Calculator FR

#3 Consider the closed curve in the xy-plane given by

$$x^2 + 2x + y^4 + 4y = 5.$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{-(x+1)}{2\left(y^3+1\right)}$.

(b) Write an equation for the line tangent to the curve at the point $(-2, 1)$.

(c) Find the coordinates of the two points on the curve where the line tangent to the curve is vertical.

(d) Is it possible for this curve to have a horizontal tangent at points where it intersects the x-axis? Explain your reasoning.

AP/IB Exam Review

Derivs Mini Lesson

<u>Non-calc FRQ #1:</u>

ⓐ $m_{tan} = f'(e^2) = \dfrac{1-\ln e^2}{(e^2)^2} = \dfrac{1-2}{e^4} = -\dfrac{1}{e^4}$

<u>point</u>: $f(e^2) = \dfrac{\ln e^2}{e^2} = \dfrac{2}{e^2} \quad \left(e^2, \dfrac{2}{e^2}\right)$

<u>tangent line</u>: $\boxed{y - \dfrac{2}{e^2} = -\dfrac{1}{e^4}\left(x - e^2\right)}$

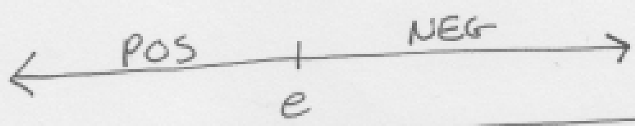
ⓑ $0 = \dfrac{1-\ln x}{x^2}$

$1 - \ln x = 0$

$\ln x = 1$

$\boxed{x = e}$

1st Deriv Test



f(x) has a relative maximum at $x=e$ because $f'(x)$ changes from positive to negative at $x=e$.

ⓒ $f''(x) = \dfrac{\left(-\frac{1}{x}\right)(x^2) - (2x)(1-\ln x)}{(x^2)^2} = \dfrac{-x - 2x + 2x\ln x}{x^4}$

$f''(x) = \dfrac{-3x + 2x\ln x}{x^4}$

$0 = \dfrac{-3x + 2x\ln x}{x^4}$

$0 = -3x + 2x\ln x$

$0 = x(-3 + 2\ln x)$

$\cancel{x = 0} \qquad -3 + 2\ln x = 0$

$\ln x = \dfrac{3}{2}$

$\boxed{x = e^{3/2}}$

The graph of f has a point of inflection at $x = e^{3/2}$ because $f''(x)$ changes sign at $x = e^{3/2}$.

ⓓ $\displaystyle\lim_{x\to 0^+} f(x) = -\infty$ or DNE

Derivs Mini Lesson

Non-calc FRQ #2:

(a) g' changes from increasing to decreasing at $x=1$ and g' changes from decreasing to increasing at $x=4$. Therefore, the points of inflection for the graph of g occur at $x=1$ and $x=4$.

(b) The only sign change of g' from positive to negative occurs at $x=2$ but you also must consider the endpoints:

Using FTC,

$$\int_{-3}^{2} g'(x)\,dx = g(x)\Big|_{-3}^{2} = g(2) - g(-3)$$

area under g'

$$-\frac{5}{2} = 5 - g(-3)$$

$$g(-3) = 7.5 \text{ or } \frac{15}{2}$$

$$\int_{2}^{7} g'(x)\,dx = g(x)\Big|_{2}^{7} = g(7) - g(2)$$

area under g'

$$-\frac{7}{2} = g(7) - 5$$

$$g(7) = \frac{3}{2} \text{ or } 1.5$$

$$g(2) = 5$$

The absolute maximum value of g on interval $-3 \le x \le 7$ is $\frac{15}{2}$. Justification is in the work above.

(c) Avg. rate of change of $g(x)$ on $-3 \le x \le 7$ $= \dfrac{g(7) - g(-3)}{7-(-3)} = \dfrac{\frac{3}{2} - \frac{15}{2}}{10} = \dfrac{\frac{-12}{2}}{10} = \dfrac{-6}{10} = \boxed{-\frac{3}{5}}$

Non-calc FRQ #2 continued:

(d) Avg. rate of change of $g'(x)$ on $-3 \le x \le 7$ $= \dfrac{g'(7) - g'(-3)}{7-(-3)} = \dfrac{1-(-4)}{10} = \dfrac{5}{10} = \boxed{\dfrac{1}{2}}$

> No, MVT does not apply because g' is not differentiable for at least one point in $-3 < x < 7$.

Non-calc FRQ #3:

(a) $x^2 + 2x + y^4 + 4y = 5$

$$2x + 2 + 4y^3\frac{dy}{dx} + 4\frac{dy}{dx} = 0$$

$$(4y^3+4)\frac{dy}{dx} = -2x - 2$$

$$\frac{dy}{dx} = \frac{-2(x+1)}{4(y^3+1)}$$

$$\boxed{\frac{dy}{dx} = \frac{-(x+1)}{2(y^3+1)}}$$

(d) Horiz. tangents occur where $\frac{dy}{dx} = 0$. So when $-(x+1) = 0$

$x = -1$

The curve crosses the x-axis where $y=0$. So,

$x^2 + 2x = 5$

$(-1)^2 + 2(-1) \overset{?}{=} 5$

$1 - 2 \overset{?}{=} 5$

$-1 \neq 5$

Therefore, No, the curve cannot have a horizontal tangent where it crosses the x-axis. Justification above.

(b) $\dfrac{dy}{dx} = \dfrac{-(-2+1)}{2(1^3+1)} = \dfrac{-(-1)}{2(2)} = \dfrac{1}{4} = m_{tan}$ point: $(-2, 1)$

> tangent line: $y - 1 = \frac{1}{4}(x+2)$

(c) Vertical tangent occur where $\frac{dy}{dx}$ is undefined. So when $2(y^3+1) = 0$

$y = -1$

$x^2 + 2x + 1 - 4 = 5$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$ → $x = -4$, $x = 2$

> points: $(-4, -1)$ and $(2, -1)$