3.6 Derivatives of Inverse Functions

Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I.

If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0$$

See below for proof in red.

or $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Deriving the formula:

If f and g are inverse functions and x is in the domain of g, then:

$$f(g(x)) = x$$ (Property of Inverse Functions)

Now, take the derivative w/ respect to x using Implicit Differentiation:

$$f'(g(x)) \cdot g'(x) = 1$$

* left side uses Chain rule

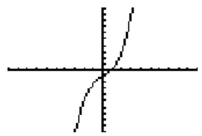
Now, solve for $g'(x)$:

$$\boxed{g'(x) = \frac{1}{f'(g(x))}}$$

Example 1) Let $f(x)=\frac{1}{4}x^3+x-1$

a. What is the value of $f^{-1}(x)$ when $x=3$?

f(x)



$\overbrace{\qquad\qquad}^{f(x)}$

X	Y1
-1	-2.25
0	-1
1	.25
2	3
3	8.75
4	19
5	35.25

X= -1

$\Rightarrow f(2)=3$

Since $f(2)=3$, then $\boxed{f^{-1}(3)=2}$

b. Hence, what is the value of $\left(f^{-1}\right)'(x)$ when $x=3$?

$$\left(f^{-1}\right)'(x)=\frac{1}{f'\left(f^{-1}(x)\right)}$$

$$\left(f^{-1}\right)'(3)=\frac{1}{f'\left(f^{-1}(3)\right)}$$

$$=\frac{1}{f'(2)}$$

$$f'(x)=\frac{3}{4}x^2+1$$

$$f'(2)=\frac{3}{4}(2)^2+1$$

$$=4$$

$$\therefore \left(f^{-1}\right)'(3)=\boxed{\frac{1}{4}}$$

Multiple Choice: AP Exam question

Example 2) Let g be the function defined by $g(x) = x^3 + x$.

If $f(x) = g^{-1}(x)$ and $f(2) = 1$, what is the value of $f'(2)$?

$g^{-1}(2) = 1$

[A] $\frac{1}{13}$ **[B]** $\frac{1}{4}$ **[C]** $\frac{7}{4}$ **[D]** 4 **[E]** 13

$$(g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))}$$

$$(g^{-1})'(2) = \frac{1}{g'(g^{-1}(2))}$$

$$= \frac{1}{g'(1)} \Rightarrow g'(x) = 3x^2 + 1$$

$$g'(1) = 3(1)^2 + 1$$

$$= 4$$

$$\therefore (g^{-1})'(2) = \frac{1}{4}$$

i.e., $f'(2) = \boxed{\frac{1}{4}}$

Derivatives of Inverse Trig Functions

$y = \arcsin x$
$y = \arccos x$
$y = \arctan x$
$y = \text{arccot } x$
$y = \text{arcsec } x$
$y = \text{arccsc } x$

These can be written as $y = \sin^{-1}x$ rather than $y = \arcsin x$

$\sin^{-1}x$ does NOT mean $\dfrac{1}{\sin x}$

Example 3: Evaluate the derivative of sin y = x

$y = \sin^{-1} x$

$\sin y = x$

$\cos y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos y}$

Recall: $\cos^2 y + \sin^2 y = 1$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ → Since $\sin y = x$ in problem

$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$ ⇒ derivative of $y = \arcsin x$

Example 4: Evaluate the derivative of cos y = x

$y = \cos^{-1} x$

$\cos y = x$

$-\sin y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{-1}{\sin y}$

Recall: $\sin^2 y + \cos^2 y = 1$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$ $\rightarrow$ since $\cos y = x$ from problem

$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}}$ $\Rightarrow$ derivative of $y = \operatorname{arc}\cos x$

MUST MEMORIZE!

$\dfrac{d}{dx}\sin^{-1}u = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$	$\dfrac{d}{dx}\tan^{-1}u = \dfrac{1}{1+u^2}\dfrac{du}{dx}$	$\dfrac{d}{dx}\sec^{-1}u = \dfrac{1}{\lvert u\rvert\sqrt{u^2-1}}\dfrac{du}{dx}$
$\dfrac{d}{dx}\cos^{-1}u = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$	$\dfrac{d}{dx}\cot^{-1}u = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$	$\dfrac{d}{dx}\csc^{-1}u = -\dfrac{1}{\lvert u\rvert\sqrt{u^2-1}}\dfrac{du}{dx}$

These formulas are on page 177 in your books.

Example 5: Find the derivative of:

$y = \sin^{-1}(1 - t)$ $\Rightarrow$ remember, this is the same as $y = \arcsin(1-t)$

Recall:

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}} \cdot u'$$

$$y' = \frac{1}{\sqrt{1-(1-t)^2}} \cdot (-1)$$

$$\boxed{y' = \frac{-1}{\sqrt{1-(1-t)^2}}}$$

Example 6: Find the derivative of:

$y = \sec^{-1}(5s)$

Recall:

$$\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|u|\sqrt{u^2-1}} \cdot u'$$

$$y' = \frac{1}{|5s|\sqrt{(5s)^2-1}} \cdot 5$$

$$\boxed{y' = \frac{5}{|5s|\sqrt{25s^2-1}}}$$

Example 7: Find the derivative of:

y = arctan (x/2)

Recall:

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{1+u^2} \cdot u'$$

$$y' = \frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \left(\frac{1}{2}\right)$$

$$y' = \frac{1}{1+\frac{x^2}{4}} \cdot \left(\frac{1}{2}\right)$$

$$y' = \frac{1}{2+\frac{x^2}{2}}$$

$$y' = \frac{1}{\frac{4+x^2}{2}}$$

$$y' = \frac{1}{1} \cdot \frac{2}{4+x^2}$$

$$\boxed{y' = \frac{2}{x^2+4}}$$

Example 8: Find the derivative of:

y = arccos (1/x)

Recall: $\frac{d}{dx}\left[\cos^{-1} u\right] = -\frac{1}{\sqrt{1-u^2}} \cdot u'$

$$y' = -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \left(-\frac{1}{x^2}\right)$$

$u = \frac{1}{x}$

$u = x^{-1}$

$u' = -x^{-2}$

$u' = -\frac{1}{x^2}$

$$\boxed{y' = \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}}$$

Example 9: Find the equation of the tangent line to the graph of

$y = \tan^{-1}x$ at $x = -2$

$$\boxed{\text{Recall: } \frac{d}{dx}\left[\tan^{-1}u\right] = \frac{1}{1+u^2} \cdot u'}$$

Slope:

$$y' = \frac{1}{1+x^2} \cdot 1$$

$$y' = \frac{1}{1+x^2}$$

$$y'(-2) = \frac{1}{1+(-2)^2}$$

$$= \frac{1}{1+4}$$

$$m_{\tan @ x=2} = \frac{1}{5}$$

Coordinate point:

$$y = \underbrace{\tan^{-1}(-2)}_{\text{use calculator in radian mode}}$$

$$y = -1.107$$

point: $(-2, -1.107)$

Tangent line @ $x=-2$ is $\boxed{y + 1.107 = \frac{1}{5}(x+2)}$

Example 10: page 180 #28

Find the derivative of $h(x) = \overbrace{\underset{f}{x^2} \cdot \underbrace{\arctan(5x)}_{g}}^{\text{product rule}}$

$f = x^2$

$f' = 2x$

$g = \arctan(5x)$

Recall: $\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{1+u^2} \cdot u'$

$g' = \frac{1}{1+(5x)^2} \cdot 5$

$g' = \frac{5}{1+25x^2}$

$h'(x) = f'g + g'f$

$h'(x) = (2x)\arctan(5x) + \left(\frac{5}{1+25x^2}\right)(x^2)$

$$\boxed{h'(x) = 2x\arctan(5x) + \frac{5x^2}{1+25x^2}}$$

Example 11: page 180 #33

Find the derivative of $h(t) = \underbrace{\sin(\arccos(t))}_{\text{Chain rule}}$

let $u = \arccos t$

$v = \sin u$

Recall: $\frac{d}{dx}[\cos^{-1} u] = -\frac{1}{\sqrt{1-u^2}} \cdot u'$

$$u' = -\frac{1}{\sqrt{1-t^2}} \cdot 1$$

$$v' = \cos u$$

$$h'(t) = u' v'$$

$$= -\frac{1}{\sqrt{1-t^2}} \cdot \cos u$$

$$= -\frac{1}{\sqrt{1-t^2}} \cdot \cancel{\cos}(\cancel{\text{arc}}\cos t)$$

$$\boxed{h'(t) = -\frac{t}{\sqrt{1-t^2}}}$$