The slope of a curve $y = f(x)$ at the point $x = a$ is

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When it exists, this limit is called "the derivative of $f$ at $a$"

f prime of a

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$f'(a)$ is read as "the derivative of $f$ with respect to $a$"

This can also be written as:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

How do derivatives work and how are they used?

**Example**: Find the derivative of the function.

$f(x) = x^3 - 6x + 3$ $\quad f'(x) = 3x^2 - 6$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - 6(x+h) + 3 - (x^3 - 6x + 3)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{6x} - 6h + \cancel{3} - \cancel{x^3} + \cancel{6x} - \cancel{3}}{h}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 6 = \boxed{3x^2 - 6}$$

$$f'(x) = \underbrace{3x^2 - 6}_{m_{tan}\ @\ \text{any } x}$$

The derivative of $f$ is the expression we use to find $m_{\tan}$ at any point.

Find $f'(1)$. $= 3(1)^2 - 6 = -3 \rightarrow m_{tan}\ @ x=1$

This is the slope of the tangent line at $x = 1$.

There are many ways to write the derivative of $y = f(x)$

| Symbol | Read as: |
|---|---|
| $f'(x)$ | "$f$ prime $x$" or "the derivative of $f$ with respect to $x$" |
| $y'$ | "$y$ prime" |
| $\frac{dy}{dx}$ | "dee why dee ecks" or "the derivative of $y$ with respect to $x$" |
| $\frac{df}{dx}$ | "dee eff dee ecks" or "the derivative of $f$ with respect to $x$" |
| $\frac{d}{dx}f(x)$ | "dee dee ecks of $f$ of ecks" (d dx of $f$ of $x$) or "the derivative of $f$ of $x$" |

# Be Careful!!!

$dx$ does not mean $d$ times $x$

$dy$ does not mean $d$ times $y$

$\frac{d}{dx}f(x)$ does NOT mean $\frac{d}{dx}$ times $f(x)$

# The Derivative Function

- The domain of $f'$ is the set of points in the domain $f$ for which the $\underbrace{\text{limit}}_{\text{deriv limit}}$ exists

- If $f'(x)$ exists, we say that $f$ has a derivative or "$f$ is differentiable"

- A function that is differentiable at every point in its domain is a differentiable function

- Functions can be differentiable at certain points

# One Sided Derivatives

A function $y = f(x)$ is differentiable on a closed interval $[a,b]$ if it has a derivative at every interior point of the interval, and if the deriv limits exist at the endpoints.

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$ the right hand derivative of *a*

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$ the left hand derivative of *b*

- Remember a function has a two sided derivative at a point iff (if and only if) the two are defined and equal at that point, but the function must be continuous at the point.

- One sided deriv limits can differ at a point and have no derivative there!

## Theorem 3.2 – The Constant Rule

The derivative of a constant function is 0. That is, if $c$ is a real number, then:

$$\frac{d}{dx}[c]=0$$

**Example 1:** Find the derivative of the following functions.

a. $y=7$

$\boxed{y'=0}$ $\frac{dy}{dx}=0$

b. $f(x)=0$

$f'(x)=0$ $\frac{df}{dx}=0$

c. $s(t)=-3$

$s'(t)=0$

d. $y=k\pi^2$, $k$ is constant

$\frac{dy}{dx}=0$

**Theorem 3.3 – The Power Rule**

If $n$ is a real number, then the function $f(x)=x^n$ is differentiable and:

$$\boxed{\frac{d}{dx}\left[x^n\right]=nx^{n-1}}$$

For $f$ to differentiable at $x=0$, $n$ must be a number such that $x^{n-1}$ is defined on an interval containing 0.

**Example 2:** Find the derivative of the following functions.

a. $f(x)=x^3$

$$\boxed{f'(x)=3x^2}$$

b. $g(x)=\sqrt[3]{x}$

$$g(x)=x^{\frac{1}{3}}$$

$$g'(x)=\frac{1}{3}x^{\frac{1}{3}-1}$$

$$=\frac{1}{3}x^{-2/3}$$

$$g'(x)=\frac{1}{3x^{2/3}}$$

c. $y=\dfrac{1}{x^2}=x^{-2}$

$$\frac{dy}{dx}=-2x^{-3}$$

$$=\boxed{\frac{-2}{x^3}}$$

d. $h(x)=x^1$

$$h'(x)=1x^{1-1}$$

$$1x^0$$

$$=\boxed{1}$$

**Example 3:** Find the slope of $f(x)=x^4$ when:

$f'(x)=4x^3$

a. $x=-1$

$f'(-1)=4(-1)^3=-4$

b. $x=0$ $f'(x)=4x^3 \rightarrow f'(0)=4(0)^3=\boxed{0}$

c. $x=1$ $f'(x)=4x^3 \longrightarrow f'(1)=4(1)^3=4$

**<u>Example 4:</u>** Find the equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

$m = -4$

$(-2, 4)$

$m_{tan} = f'(x) = 2x$

$m_{tan} = f'(-2) = 2(-2) = -4$
$@x=-2$

$y - 4 = -4(x+2)$