8.7 Finding Area and Arc Length in Polar Coordinates

**Objectives**
- Students will calculate areas and arc lengths in polar coordinates
- Students will find areas in overlapping polar graphs
- Students will verify and justify solutions.
Polar Coordinates

Pole: Origin \((0,0) = (0, \theta)\) \(\theta \in \mathbb{R}\)

Polar Axis: Positive x-axis

Point \((r, \theta) = (r, \theta \pm 2k\pi) = (-r, \theta \pm k\pi)\)

Where \(k \in \mathbb{Z}\)

\(r = \) radius; i.e. distance from the pole
\(\theta = \) angle measured counterclockwise from polar axis
Polar $\rightarrow$ Rectangular

\[
\begin{align*}
X &= r \cos \theta \\
Y &= r \sin \theta
\end{align*}
\]

Rectangular $\rightarrow$ Polar

\[
\begin{align*}
x^2 + y^2 &= r^2 \\
r &= \pm \sqrt{x^2 + y^2} \\
\tan \theta &= \frac{y}{x} \rightarrow \theta &= \tan^{-1} \frac{y}{x}
\end{align*}
\]
1. Convert to rectangular:
\[(6, \pi/6) \rightarrow x = 6 \cos \frac{\pi}{6} = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3} \Rightarrow (x, y) = (3\sqrt{3}, 3)\]
\[y = 6 \sin \frac{\pi}{6} = 6 \cdot \frac{1}{2} = 3\]

2. Convert \((2, -2)\) to polar:
\[r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}\]
\[\tan \theta = -2/2 = -1 \Rightarrow \theta = -\pi/4\]

3. Convert \(r = 4 \sin \theta\) to rectangular:
\[x^2 + y^2 = r^2\]
\[x^2 + y^2 = 4y\]
\[x^2 + y^2 - 4y = 0 + 4\]
\[x^2 + (y - 2)^2 = 4\]

Circle center \((0, 2)\)
\[r = 2\]
1. a) Find the arc length of a sector of central angle $\pi/6$ of a circle with radius 4. (hint: set up a proportion)

\[ L = \frac{\pi}{6} \times 4 \]

b) find the area of the sector

\[ A = \frac{1}{2} \times \frac{\pi}{6} \times 4^2 = \frac{1}{2} \times \frac{\pi}{6} \times 16 = \frac{4\pi}{3} \]

2. a) Find the arc length of a sector of with central angle $3\pi/2$ of a circle with radius 10.

\[ L = \frac{3\pi}{2} \times 10 = 15\pi \]

b) find the area of the sector.

\[ A = \frac{1}{2} \times \frac{3\pi}{2} \times 10^2 = \frac{1}{2} \times \frac{3\pi}{2} \times 100 = 75\pi \]
Curves in polar coordinates

Graph each curve with the graphing calculator in polar mode, then use the trace feature to see how the curve gets drawn as $\theta$ increases.

\[ r = 1 - 2\cos\theta \]

Find the points of intersection between the two curves. (algebraically, then use graphing calculator.)

For $r = 1 - 2\cos\theta$

1. $-2\cos\theta = 1$
2. $\cos\theta = 0$

For $r = 1$

1. $\cos\theta = 1$
2. $\theta = 0$
Show the $dx$, $dy$, and $dl$ on the graph and write an expression for $dl$ in rectangular form and in parametric form.

**Rectangular**

$$dl = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

**Parametric**

$$dl = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2} dx$$

Calculate the length of $y = 2\cos(x)$ from $x = 0$ to $x = \pi$.

$$L = \int_{0}^{\pi} \sqrt{1 + (\cos(x))^2} dx$$

$$y = 2\sin(x)$$
We can also find arclength when using equations in polar mode:

\[ dl = \sqrt{dr^2 + (r\,d\theta)^2} \]
Arc Length of a Curve in Polar Coordinates

\[ dL = \sqrt{\left( \frac{dr}{d\theta} \right)^2 + r^2} \ d\theta \]

\[ dL = \sqrt{\left( \frac{dr}{d\theta} \right)^2 + r^2} \ d\theta \]

\[ L = \frac{1}{2} \int_{a}^{b} \sqrt{dr^2 + (r d\theta)^2} \]

\[ L = \frac{1}{2} \int_{a}^{b} \sqrt{\left( \frac{dr}{d\theta} \right)^2 + r^2} \ d\theta \]
Calculate the length of one full loop of the cardioid.

\[ r = 2 + 2 \sin \theta \]
\[ \frac{dr}{d\theta} = 2 \cos \theta \]

\[ r_1 = 2 + 2 \sin \theta \]
\[ r_2 = \frac{1}{2} \left| \frac{dr}{d\theta} \right| = 2 \cos \theta \]

\[ L = 2\pi \int_{0}^{\pi} \sqrt{(r_1)^2 + (r_2)^2} \, d\theta \]

\[ L = 2\pi \int_{0}^{\pi} \sqrt{(2 + 2 \sin \theta)^2 + (2 \cos \theta)^2} \, d\theta \]

\[ L = 2\pi \int_{0}^{\pi} \sqrt{r_1^2 + r_2^2} \, d\theta \]

\[ L = 2\pi \int_{0}^{\pi} (r_1^2 + r_2^2)^{1/2} \, d\theta \]
4. Find the length of the limaçon $r = 1 + 3\sin\theta$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta = 19.376 \quad \text{or} \quad 19.377$$
Areas Under Curves in Rectangular Coordinates

Represent \( \int f(x) \, dx \) as a Midpoint Riemann Sum with 8 equal subintervals.

Write the Definition of Definite Integral

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f \left( \frac{b-a}{n} \cdot i + \frac{a+b}{2} \right) \frac{b-a}{n}
\]

Explain the meaning of \( dA \) and \( dx \) in the context of finding an area under a curve by integration.
8.7 Notes

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Polar to Rectangular:

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

What do each of the following tell us?

\[ \frac{dy}{d \theta} > 0 \Rightarrow \text{do } \theta \text{ inc, it gets further from pole} \]
\[ \frac{dy}{d \theta} < 0 \Rightarrow \text{do } \theta \text{ dec, it gets closer to pole} \]

Area Enclosed by Polar Graphs:

\[ \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta \]

\[ A = \frac{2\pi}{12} \int_0^{\pi/2} \frac{1}{2} r^2 d\theta \]

\[ A = \frac{2\pi}{12} \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \frac{1}{2} \pi \left( \frac{\pi}{8} \right) = \frac{1}{16} \pi \]

\[ A = \frac{2\pi}{12} \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \frac{1}{2} \pi \left( \frac{\pi}{8} \right) = \frac{1}{16} \pi \]
\[ dA = \frac{1}{2} r^2 d\theta \]
\[ A = \int_{a}^{b} \frac{1}{2} r^2 d\theta \]
5. The graph below shows the limacon \( r = 1 + 3 \sin \theta \)

a. Find the area of the region inside the inner loop.

\[
\frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta = A
\]

\( A = \frac{1}{2} \int_{\theta_1}^{\theta_2} \left(1 + 3 \sin \theta \right)^2 \, d\theta \)

b. Find the area of the region between the outer loop and the inner loop.

\[
\frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta = \text{Area}
\]

\[
\frac{1}{2} \int_{\theta_1}^{\theta_2} \left(1 + 3 \sin \theta \right)^2 \, d\theta = 14.75 \text{ units}^2
\]

\[
\int_{\theta_1}^{\theta_2} r \, d\theta = 12.23 \text{ units}^2
\]
6. \[ L = 2 \int_{\theta_1}^{\pi/2} \sqrt{(4 \sin 2\theta)^2 + (4 \cos 2\theta)^2} \, d\theta \]

(a) Find the length of one petal of the three petal rose traced by \( r = 4 \cos 3\theta \).

(b) Find the area enclosed by all three petals.

Area of

0 3 petals = 6 \( \text{Area of 1 petal} \),

\[ \text{Area} = 6 \int_{\pi/6}^{\pi/3} \frac{1}{2} r^2 \, d\theta \]

= 12.544
7. a) Find the arc length of one full loop of $r = 5 + 4\cos\theta$, and 
b) Find the area enclosed by the limacon $r = 5 + 4\cos\theta$
a) 
$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$
$$= \approx 36.0878$$
b) 
$$A = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta = \approx 103.6725$$