Review-warm-up

Function composition is the application of one function to another. The output of one function becomes the input of another. The operation is denotes by the symbol $(f \circ g)(x)$ or $f(g(x))$

Example 1: Let $f(x) = x + 1$ and $g(x) = 2x + 3$
Calculate $f(g(-1))$

Method 1

$f(g(x)) = g(x) + 1$
$= 2x + 3 + 1$
$= 2x + 4$
$f(g(-1)) = 2(-1) + 4$
$= 2$

Method 2

$g(-1) = 2(-1) + 3$
$= 1$
$f(1) = 1 + 1 = 2$

Example 2: $f(x) = 9x$ and $g(x) = \sqrt{x}$

find $g(f(4))$ in two different ways.

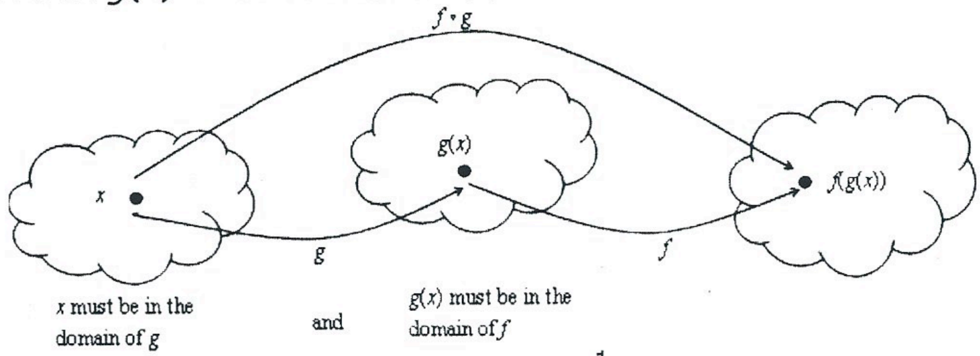
method 1.

$g(f(x)) = \sqrt{f(x)}$
$= \sqrt{9x}$
$= 3\sqrt{x}$
$g(f(4)) = 3\sqrt{4}$
$= 6$

method 2.

$f(4) = 9(4) = 36$
$g(36) = \sqrt{36}$
$= 6$

The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.



Example 5: Given $f(x) = f(x) = \frac{1}{x+2}$ and $g(x) = g(x) = \frac{4}{x-1}$

What is the domain of $f(g(x))$?

$D_g : \{x \mid x \neq 1\}$

$f(g(x)) = \frac{1}{g(x)+2}$

$g(x) \neq -2$

$\frac{4}{x-1} \neq -2 \rightarrow 4 \neq -2x+2 \rightarrow x \neq -1$

$D_{f \circ g} : \{x \mid x \neq -1, 1\}$

4.1 One-to-One and Inverse Functions

Learning Objectives

In Section 4.1 you will learn how to:

- ☐ **A.** Identify one-to-one functions
- ☐ **B.** Explore inverse functions using ordered pairs
- ☐ **C.** Find inverse functions using an algebraic method
- ☐ **D.** Graph a function and its inverse
- ☐ **E.** Solve applications of inverse functions

The power and importance of exponential and logarithmic functions would be hard to overstate. From ecology and economics to environmental studies and atomic research, we simply couldn't do without them. While many important applications may be of interest only to scientists or investment firms, their use actually extends in many directions, and often serves to broaden our underatnding of history and civilization. For instance, using bits of organic material from the immediate area of the famous Stonehenge site (in southern England), scientists were able to estimate the date that the area was last inhabited, even though it was thousands of years ago. This application appears as Exercise 62 in Section 4.5

One-to-One Functions

A function f is one-to-one if every element in the range corresponds to only one element of the domain.

In symbols, if $f(x_1) = f(x_2)$ then $x_1 = x_2$, or
if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

From this definition we note the graph of a one-to-one function must not only pass a vertical line test (to show each x corresponds to only one y), but also pass a **horizontal line test** (to show each y corresponds to only one x).

Horizontal Line Test

If every horizontal line intersects the graph of a function in at most one point, the function is one-to-one.

Notice the graph of $y = 2x$ (Figure 4.3) passes the horizontal line test, while the graph of $y = x^2$ (Figure 4.4) does not.

Figure 4.3

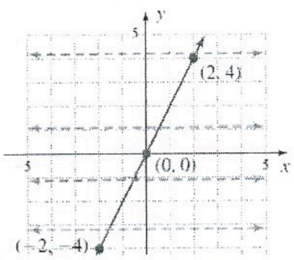
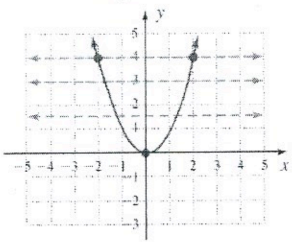


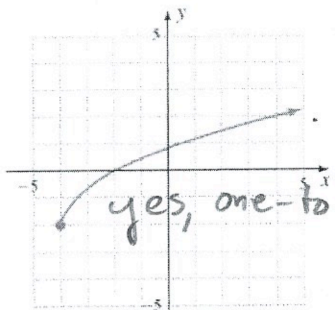
Figure 4.4



Identifying One-to-One Functions

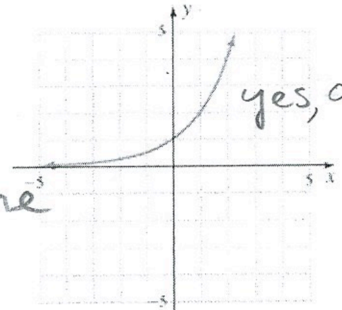
Use the horizontal line test to determine whether each graph is the graph of a one-to-one function.

a.



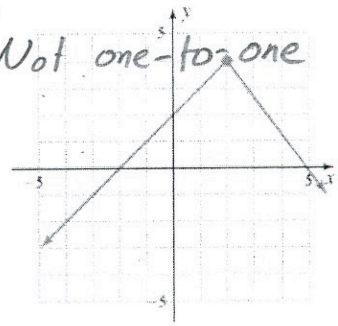
yes, one-to-one

b.



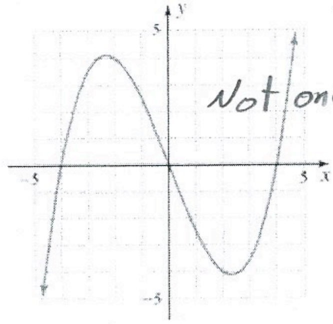
yes, one-to-one

c.



Not one-to-one

d.



Not one-to-one

Inverse Functions

If f is a one-to-one function with ordered pairs (a, b),

1. $f^{-1}(x)$ is a one-to-one function with ordered pairs (b, a).
2. The range of f will be the domain of $f^{-1}(x)$.
3. The domain of f will be the range of $f^{-1}(x)$.

Finding the Inverse of a Function

Find the inverse of each one-to-one function given:

a. $f(x) = \{(-4, 13), (-1, 7), (0, 5), (2, 1), (5, -5), (8, -11)\}$

b. $p(x) = -3x + 2$

a. $f^{-1}(x) = \{(13,-4), (7,-1), (5,0), (1,2), (-5,5), (-11,8)\}$

b. Undo the operation in reverse order:
subtract 2 then $\div$ by -3

$\Rightarrow \quad p^{-1}(x) = \dfrac{x-2}{-3}$

C. Finding Inverse Functions Using an Algebraic Method

The fact that interchanging x- and y-values helps determine an inverse function can be generalized to develop an **algebraic method** for finding inverses. Instead of interchanging *specific* x- and y-values, we actually interchange the x- and y-*variables*, then solve the equation for y. The process is summarized here.

Finding an Inverse Function

1. Use y instead of $f(x)$.
2. Interchange x and y.
3. Solve the equation for y.
4. The result gives the inverse function: substitute $f^{-1}(x)$ for y.

Finding Inverse Functions Algebraically

Use the algebraic method to find the inverse function for

a. $f(x) = \sqrt[3]{x+5}$

$x = \sqrt[3]{y+5}$

$x^3 = y+5$

$y = x^3 - 5$

$f^{-1}(x) = x^3 - 5$

b. $g(x) = \dfrac{2x}{x+1}$

$x = \dfrac{2y}{y+1}$

$xy + x = 2y$

$x = 2y - xy$

$x = y(2-x)$

$y = \dfrac{x}{2-x}$

$f^{-1}(x) = \dfrac{x}{2-x}$

Find the inverse of the function $y = \dfrac{2x+1}{x-1}$

$x = \dfrac{2y+1}{y-1}$

$xy - x = 2y + 1$

$-x - 1 = 2y - xy$

$-x - 1 = y(2-x)$

$y = \dfrac{(-x-1)}{2-x}$

or

$xy - x = 2y + 1$

$xy - 2y = 1 + x$

$y(x-2) = 1 + x$

$y = \dfrac{1+x}{x-2}$

$f^{-1}(x) = \dfrac{-x-1}{2-x} = f^{-1}(x) = \dfrac{1+x}{x-2}$

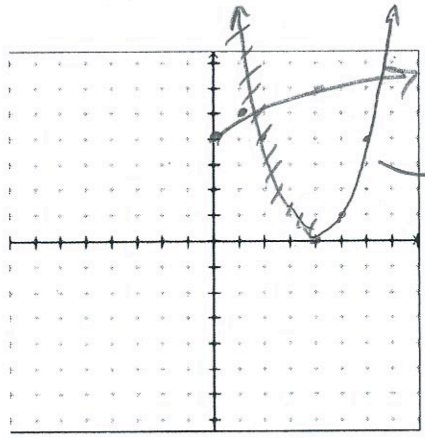
Restricting the Domain to Create a One-to-One Function

Given $f(x) = (x - 4)^2$, restrict the domain to create a one-to-one function, then find $f^{-1}(x)$. State the domain and range of both resulting functions.

Restrict Domain to

$D_f: x \in [4, \infty)$

$\rightarrow R_f: y \in [0, \infty)$



$x = (y - 4)^2$

$\pm\sqrt{x} = y - 4$

$y = \pm\sqrt{x} + 4$

$R_{f^{-1}}: y \in [4, \infty) \rightarrow f^{-1}(x) = \sqrt{x} + 4$

$D_{f^{-1}}: x \in [0, \infty)$

Verifying Inverse Functions

If f is a one-to-one function, then the function f^{-1} exists, where

$$(f \circ f^{-1})(x) = x \quad \text{and} \quad (f^{-1} \circ f)(x) = x$$

We denote the inverse of a function f as f^{-1}. The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

ex. $f(x) = \{(1,2)\}$ $\quad f(f^{-1}(2)) =$

$f^{-1}(x) = \{(2,1)\}$ $\quad f(1) = 2$

The composition of a function and its inverse is always x. That is to say

each function undoes the operation of the other.

Verify that the inverse of $f(x)=\dfrac{1}{x-1}$ is $f^{-1}(x)=\dfrac{1}{x}+1$

$$f(f^{-1}(x))=\frac{1}{f^{-1}(x)-1}=\frac{1}{\frac{1}{x}+1-1}=\frac{1}{\frac{1}{x}}=x$$

and

$$f^{-1}(f(x))=\frac{1}{f(x)}+1=\frac{1}{\frac{1}{x-1}}+1=x-1+1=x$$

$D_f: x \in [-2,\infty)$

$R_f: y \in [0,\infty)$

Finding and Verifying an Inverse Function

Use the algebraic method to find the inverse function for $f(x)=\sqrt{x+2}$. Then verify the inverse you found is correct.

$$x=\sqrt{y+2} \rightarrow x^2=y+2 \rightarrow f^{-1}(x)=x^2-2$$

$D_{f^{-1}}: x \in [0,\infty)$

$R_{f^{-1}}: y \in [-2,\infty)$

$$f(f^{-1}(x))=\sqrt{f^{-1}(x)+2}=\sqrt{x^2-2+2}=\sqrt{x^2}=x$$

$$f^{-1}(f(x))=f(x)^2-2=(\sqrt{x+2})^2-2=x+2-2=x$$

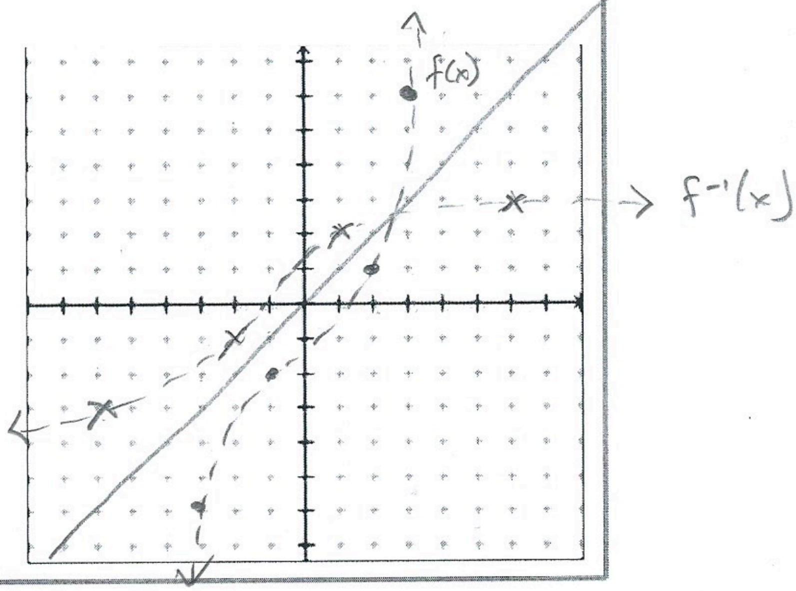
Graphing a function and its inverse:

Find the inverse of the function given and graph both functions in two different colors.

x	$f(x)$
3	6
2	1
–1	–2
–3	–6

x	$f^{-1}(x)$
6	3
1	2
–2	–1
–6	–3

Note Symmetry to $y=x$

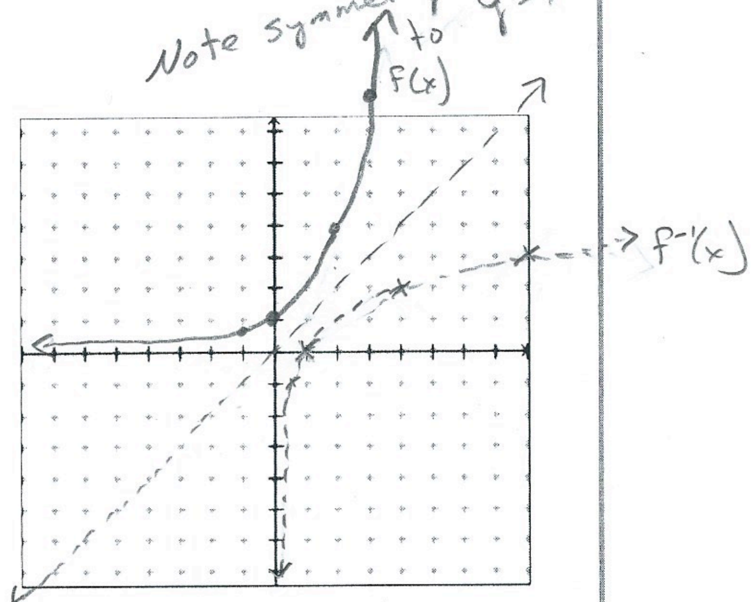


Find the inverse of the function given and graph both functions in two different colors.

x	$f(x)$
3	8
0	1
–1	½
2	4

x	$f^{-1}(x)$
8	3
1	0
½	–1
4	2

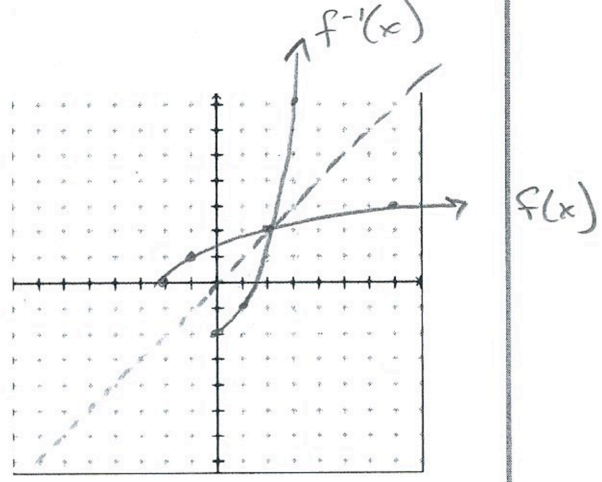
Note symmetry to $y=x$



Graphing a Function and Its Inverse

In Example 5, we found the inverse function for $f(x) = \sqrt{x+2}$ was $f^{-1}(x) = x^2 - 2, x \geq 0$. Graph these functions on the same axes and comment on how the graphs are related.

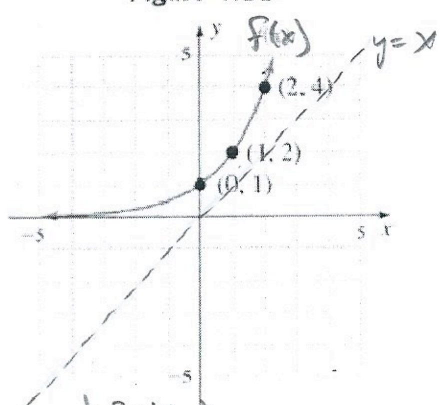
The function and its inverse are symmetric to the line $y = x$



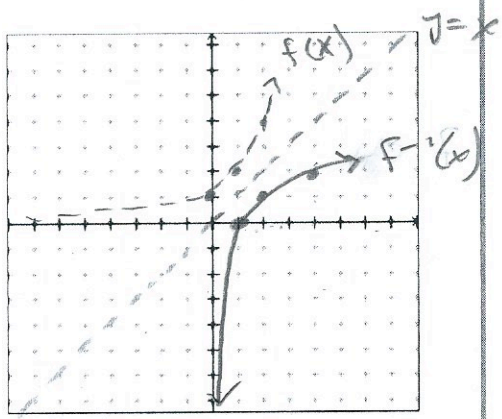
Graphing a Function and Its Inverse

Given the graph shown in Figure 4.11. ~~of the inverse function.~~ → Draw the graph of the inverse function:

Figure 4.11



x	$f^{-1}(x)$
1	0
2	1
4	2

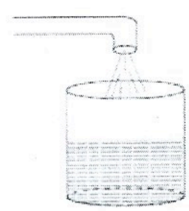


SUMMARY

Functions and Inverse Functions

1. If the graph of a function passes the horizontal line test, the function is one-to-one.
2. If a function f is one-to-one, the function f^{-1} exists.
3. The domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.
4. For a function f and its inverse f^{-1}, $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.
5. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.

Using Volume to Understand Inverse Functions



The volume of an equipoise cylinder (height equal to diameter) is given by $v(x) = 2\pi x^3$ (since $h = d = 2r$), where $v(x)$ represents the volume in units cubed and x represents the radius of the cylinder.

a. Find the volume of such a cylinder if $x = 10$ ft.

b. Find $v^{-1}(x)$, and discuss what the input and output variables represent.

c. If a volume of 1024π ft^3 is required, which formula would be easier to use to find the radius? What is this radius?

a. $v(10) = 2\pi\, 10^3 = 2000\pi$ ft^3

b. $x = 2\pi y^3 \rightarrow y^3 = \frac{x}{2\pi} \rightarrow f^{-1}(x) = \sqrt[3]{\frac{x}{2\pi}}$

$x =$ volume
$f^{-1}(x) =$ the radius that holds this volume.

C. use the Inverse function:

$f^{-1}(1024\pi) = \sqrt[3]{\frac{1024\pi}{2\pi}} = \sqrt[3]{512}$

$= 8$ ft

8 ft is the radius when $v(x) = 1024\pi$ ft^3